

# Geometry and topology of two kinds of extreme Reissner-Nordström-anti-de Sitter black holes

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## Abstract

Different geometrical and topological properties have been shown for two kinds of extreme Reissner-Nordström-anti-de Sitter black holes. The relationship between the geometrical properties and the intrinsic thermodynamical properties has been made explicit.

PACS number(s): 04.70.Dy, 04.20.Gz, 04.62.+v.

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The study of the extreme black hole (EBH) has been stimulated since the discovery [1,2] that the four-dimensional (4D) Reissner-Nordström (RN) EBH is a different object from its nonextreme counterpart owing to its drastically different topological properties and peculiar zero entropy regardless of its nonzero horizon area. However, these results met some challenges subsequently. Starting with the grand canonical ensemble, it was argued [3-5] that in a finite size cavity a 4D RN non-extreme black hole (NEBH) can approach the extreme state as closely as one likes and the geometrical and topological properties are still of nonextreme sectors. Bekenstein-Hawking formula is believed to hold for RN EBH entropy description, which is also supported by state-counting calculations of certain extreme and near-extreme black holes in string theory, see [6] for a review. These different results indicate that EBHs have a controversial role in black hole thermodynamics and topologies and require special care.

Comparing [1,2] and [3-5], it seems that the clash comes from two different treatments: one refers to Hawking's treatment by starting with the original EBH [1,2] and the other Zaslavskii's treatment by first taking the boundary limit and then the extreme limit to get the EBH from its nonextreme counterpart [3-5]. Applying these two treatments, it was found that two different topological objects represented by different Euler characteristics exist for 4D RN EBH, charged dilaton EBH [7], Kerr EBH [8] as well as two-dimensional (2D) EBHs [7,9]. Drastically different intrinsic thermodynamical properties have also been displayed for 4D Kerr EBH [8] and 2D EBHs [9,10] due to these different treatments. Based upon these results it was suggested that there maybe two kinds of EBHs in nature: the first kind suggested by Hawking et al with the extreme topology and zero entropy, which can only be formed by pair

creation in the early universe, while the second kind, suggested by Zaslavskii, has the topology of the nonextreme sector and the entropy is still described by the Bekenstein-Hawking formula, which can be developed from its nonextreme counterpart through second order phase transition [11-13]. This speculation has been further confirmed recently in a Hamiltonian framework [14] and the grand canonical ensemble [15] as well as canonical ensemble [16] formulation for RN anti-de Sitter (AdS) black hole, where the Bekenstein-Hawking entropy and zero entropy have been found again for RN AdS EBHs. However the clear pictures of geometry and topology for two kinds of RN AdS EBHs have not been presented therein.

The study of RN AdS black holes is appealing. This is not only because it is a standard example to study the AdS/CFT correspondence [17], but also because some striking resemblance of the RN AdS phase structure to that of the Van de Waals-Maxwell liquid-gas system has been observed and some classical critical phenomena has also been uncovered [18,19]. It is of interest to investigate the geometrical and topological properties of RN AdS EBHs and their relation to the EBHs' intrinsic thermodynamics. We hope that detailed understanding of geometrical and topological properties in RN AdS EBHs will help us to get a clear picture of the phase transition in RN AdS black holes. This is the motivation of the present paper.

The RN black hole solution of Einstein's equations in free space with a negative cosmological constant  $\Lambda = -3/l^2$  is given by

$$ds^2 = -h dt^2 + h^{-1} dr^2 + r^2 d\Omega^2, A = Q/r dt, \quad (1)$$

with

$$h = 1 - \frac{r_+}{r} - \frac{r_+^3}{l^2 r} - \frac{Q^2}{r_+ r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}. \quad (2)$$

The asymptotic form of this spacetime is AdS. There is an outer horizon located at  $r = r_+$ . The mass of the black hole is

$$M = \frac{1}{2}(r_+ + \frac{r_+^3}{l^2} + \frac{Q^2}{r_+}). \quad (3)$$

The Hawking temperature is given by the expression

$$T_H = \frac{1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2}}{4\pi r_+} \quad (4)$$

and the potential by

$$\phi = \frac{Q}{r_+} \quad (5)$$

In the extreme case  $r_+, Q$  satisfy the relation

$$1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2} = 0. \quad (6)$$

Making use of the approach developed in [20], for a grand canonical ensemble the RN AdS black hole is considered in a cavity with radius  $r_B$ , and the local temperature on the cavity is

$$T = \frac{T_H}{[h(r_B)]^{1/2}} = \frac{T_H}{[(1 - \frac{Q^2}{r_+ r_B} + \frac{3r_B^2}{l^2}) - \frac{r_+}{r_B}(1 - \frac{Q^2}{r_B r_+} + \frac{r_+^2}{l^2} + \frac{2r_B^3}{l^2 r_+})]^{1/2}}. \quad (7)$$

When the black hole approaches the extreme state,  $T_H \rightarrow 0$ , the simplest choice is to take  $T \rightarrow 0$ . One might refer to the third law of thermodynamics to argue that the EBH cannot be achieved.

However, it is interesting to point out that although  $T_H \rightarrow 0$ , the square root in (7) tends to zero as well if we take  $r_+ \rightarrow r_B$ . Thus the extreme state of RN AdS black hole with nonzero local temperature can be achieved with no contradiction with the third law. In the grand canonical ensemble actually

only the temperature on the boundary has physical meaning, whereas  $T_H$  can always be rescaled without changing observable quantities [21].

Now it is of interest to study the proper distance between a horizon and any fixed point. Different behaviors of proper distances obtained in [1] and [3] for RN EBHs are cognitive roots to qualitatively different two types of EBHs. In our metric (1), this is the quantity

$$L = \int_{r_+}^{r_B} \frac{dr}{\sqrt{h(r)}} = \int_{r_+}^{r_B} \frac{r dr}{\sqrt{\frac{r^4}{l^2} + r^2 - (r_+ + \frac{r_+^3}{l^2} + \frac{Q^2}{r_+})r + Q^2}} \quad (8)$$

To study the behavior of  $L$  for the RN AdS EBH, we have two limits to deal with, one is the extreme limit and the other is the boundary limit. To perform the limit procedures, we may take  $Q^2 = (1 + \frac{3r_+^2}{l^2})r_+^2 + \epsilon, \epsilon \rightarrow 0^+$  and  $r_B = r_+ + \eta, \eta \rightarrow 0^+$ , where  $\epsilon, \eta$  are infinitesimal quantities with different orders of magnitude. For Hawking's treatment, starting with the original RN AdS EBH, the proper distance can be written as

$$L_1 = \int_{r_+}^{r_B} \frac{r dr}{\sqrt{(r - r_+)^2 + (r^4 - 4r_+^3 r + 3r_+^4)/l^2}} \quad (9)$$

By taking the boundary limit  $r_+ \rightarrow r_B$ , this quantity diverges (Fig.1). However for Zaslavskii's treatment by first taking the boundary limit and then the extreme limit, we can evaluate the behavior of the proper distance  $L_2$  mathematically by taking  $\eta = \epsilon^p, p > 1$ , it leads to a finite value as shown in Fig.2. These results are consistent with those obtained in RN cases [1,3].

The characteristic of the proper distance can also be derived by studying the limiting geometry of the extreme RN AdS black holes. The Euclidean black hole metric can be written as

$$ds^2 = h d\tau^2 + h^{-1} dr^2 + r^2 d\Omega^2 \quad (10)$$

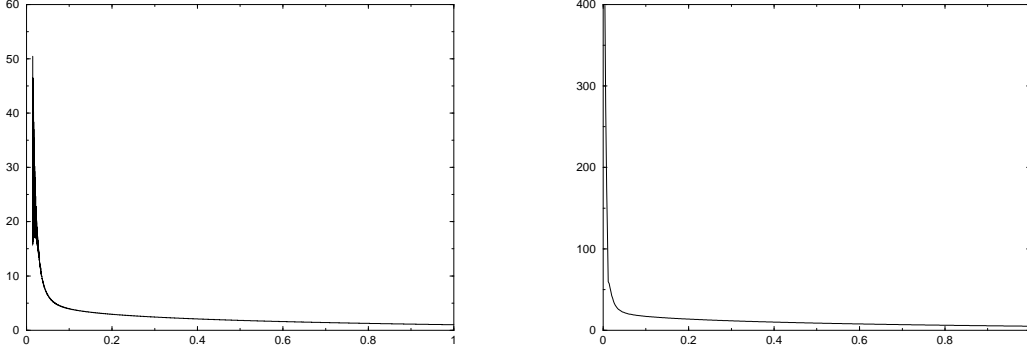


Figure 1: Behaviour of  $L_1$  with  $\epsilon$ , first for a parameter  $l = 1$ , second diagram for  $l = 5$ . We chose the parameter  $r_+ = 10$ .

The Euclidean time takes its value in the range  $0 \leq \tau \leq T_H^{-1}$ . Introducing a new variable  $\tau_1 = 2\pi T_H \tau$  and  $0 \leq \tau_1 \leq 2\pi$ , we have

$$ds^2 = (\beta/2\pi)^2 d\tau_1^2 + dl^2 + r^2 d\Omega^2 \quad (11)$$

where  $\beta[r(l)] = \beta_H[h(r)]^{1/2}$  is the inverse local temperature at an arbitrary point  $r_+ \leq r \leq r_B$ .  $l$  is the proper distance between  $r_+$  and  $r$ . We choose the coordinate according to

$$r - r_+ = 4\pi T_H b^{-1} \sinh^2(x/2), \quad b = h''(r_+)/2. \quad (12)$$

In the limit  $r_+ \rightarrow r_B$ , where the hole tends to occupy the entire cavity, the region  $r_+ \leq r \leq r_B$  shrinks and we can expand  $h(r)$  in a power series  $r - r_+$ . After substituting Eqs.(7,4) into (11) and taking the extreme limit in the end,

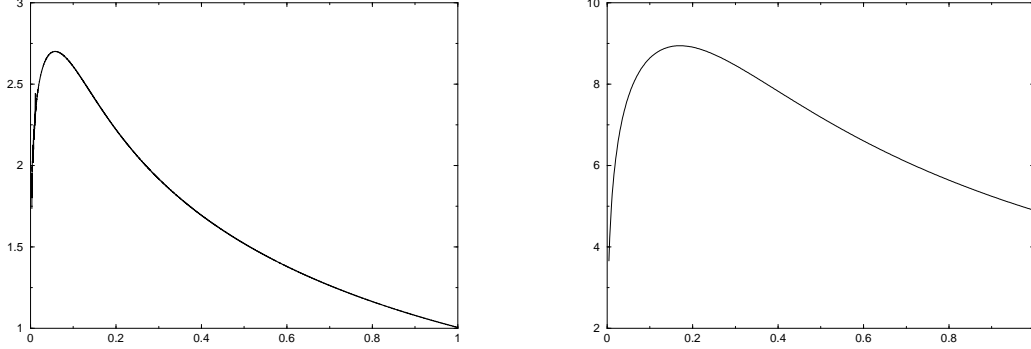


Figure 2: Behaviour of  $L_2$  with  $\epsilon$ , first for a parameter  $l = 1$ , second diagram for  $l = 5$ . We chose the parameter  $r_+ = 10$ .

we obtain

$$ds^2 = r_B^2 \left[ \frac{1}{1 + 6r_B^2/l^2} (d\tau_1^2 \sinh^2 x + dx^2) + d\Omega^2 \right] \quad (13)$$

In the Lorenzian version Eq(13) can be expressed as

$$ds^2 = r_B^2 \left[ \frac{1}{1 + 6r_B^2/l^2} (-dt^2 \sinh^2 x + dx^2) + d\Omega^2 \right] \quad (14)$$

This is the extension of the Bertotti-Robinson spacetime [22]. Taking  $\Lambda = -3/l^2 \rightarrow 0$ , it returns to the RN result in [4].

Now we are in a position to discuss the properties of the metric (14). In the extreme case the horizon is at  $x = 0$ , as can be seen by noting that the metric in (14) degenerates at  $x = 0$ . The proper radial distance between the horizon and any other point is finite, which is in agreement with the numerical result exhibited in Fig.2.

Now we turn to concentrate our attention on the original RN AdS EBH.

This black hole satisfies Eq(6) at the very beginning. We put it in a cavity with the boundary  $r_B$ . The expression of  $h(r)$  now is

$$h(r) = (1 - r_+/r)^2 + (r^2 - 4r_+^3/r + 3r_+^4/r^2)/l^2 \quad (15)$$

Expanding the metric coefficients near  $r = r_+$  and introducing  $r - r_+ = r_B \rho^{-1}$  [4,8], we obtain

$$ds^2 = r_B^2 \rho^{-2} [-(1 + 6r_B^2/l^2)dt_1^2 + d\rho^2 + \rho^2 d\Omega^2] \quad (16)$$

in the limit  $r_+ \rightarrow r_B$ . Considering the vanishing cosmological constant, it again boils down to (8) in the paper [4] for RN original EBH case.

Using  $r_B^2 \rho^{-2} = 0$  to determine the horizon, we find that the horizon is infinitely far away ( $\rho = \infty$ ). Therefore, the proper distance between the horizon and any other point at finite  $\rho$  is infinite, which agrees to what is shown in Fig.1.

Different behaviors of the proper distances directly relate to different Euler characteristics corresponding to the topology for RN AdS EBHs. The Euclidean metric can be rewritten as

$$ds^2 = e^{2u(r)} d\tau^2 + e^{-2u(r)} dr^2 + R^2 d\Omega^2. \quad (17)$$

From the Gauss-Bonnet (GB) theorem and the boundary condition, the Euler characteristic  $\chi$  takes the form [23,7]:

$$\chi = \frac{\beta_H}{2\pi} [(2u'e^{2u})(1 - e^{2u}R'^2)]_{r_+}^{r_0} \quad (18)$$

where  $\beta_H = 4\pi[(e^{2u})'_{r=r_+}]^{-1}$ , and  $e^{2u(r)} = h(r)$ ,  $R^2 = r^2$  in our case.

For Zaslavskii's treatment to obtain EBH, namely, by first taking the boundary limit and then extreme limit, we have

$$\begin{aligned} [(e^{2u}R'^2)|_{r=r_+=r_B}]_{extr} &= [1 - r_+/r - r_+^3/(l^2r) - Q^2/(r_+r) + Q^2/r^2 + r^2/l^2]_{r=r_+=r_B} |_{extr} \\ &= [1 - r_B/r_B - r_B^3/(l^2r_B) - Q^2/r_B^2 + Q^2/r_B^2 + r_B^2/l^2]_{extr} = 0 \end{aligned} \quad (19)$$



Since the horizon locates at a finite position,  $\beta_H$  can be fixed, therefore the Euler characteristic  $\chi = \frac{4\pi}{2\pi}(\frac{h'}{h})_{r=r_+=r_B} |_{extr} = 2$ . This is the same result as that of the NEBH.

However, for Hawking's treatment, starting with the original EBH, since the horizon is infinitely far away,  $\beta_H$  is not fixed. Under such a condition we could get any value from the volume integral in the GB action supplemented by the outer boundary contribution. As done for 4D RN black hole as well as 4D charged dilaton black hole [23], the only way to get a unique result is to add the inner boundary  $r_0 = r_+ + \epsilon$  and set  $\epsilon \rightarrow 0$  in the end of calculation. This will lead unambiguously to  $\chi = 0$ , which differs drastically from that of the NEBH result.

By means of the relation between the Euler characteristic and the entropy first derived for NEBH [24] and later applied to EBHs [7,8]

$$S = \frac{A}{8}\chi \quad (20)$$

and the different Euler characteristics obtained above for two different treatments, naturally we can naturally conclude that the RN AdS EBH with nonextreme topology has the entropy of  $A/4$ , while for the RN AdS EBH with extreme topology, zero entropy emerges. These results can be used to explain the intrinsic thermodynamical results obtained in [14-16].

In summary, we have shown that in the grand canonical ensemble, the 4D RN AdS black hole can approach the extreme state at nonzero temperature. The proper distance and geometrical properties of the EBH developed from its NEBH counterpart and the original EBH have also been exhibited. From limiting metrics and Euler characteristics, we found that these two EBHs are in different topological sectors, nonextreme and extreme configurations, respectively. Due to different spacetime topology of these two kinds of EBHs, it

is easy to understand different intrinsic thermodynamical properties,  $S = A/4$  for EBH of NEBH topology and  $S = 0$  for EBH of EBH topology. These results also affect the understanding of the phase transition of RN AdS black holes. It has been shown that a phase transition exists for a lot of black holes at the extreme limit [11-13]. Recently some classical critical phenomena has also been uncovered for RN AdS black holes [18-19]. It would be reasonable to understand that phase transition can happen between the RN AdS NEBH and EBH with a nonextreme topological configuration. The RN AdS EBH with extreme topological configuration is a quite different object as compared to the NEBH and thus a phase transition cannot happen between them.

ACKNOWLEDGEMENT: This work was partically supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPQ). B. Wang would also like to acknowledge the support given by Shanghai Science and Technology Commission.

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